THE EFFECTIVE PATH LENGTH THROUGH A VERTICAL COSMIC-RAY TELESCOPE.*

BY

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I. INTRODUCTION.

One of the most widely used methods for investigating the properties of the cosmic radiation involves measurement of their absorption in matter. By means of coincidence-counter telescopes (see Fig. 1) which record an event only when every member of a train of Geiger-Mueller counters is discharged simultaneously by an ionizing particle, it is possible to select rays which have followed paths lying within specified limits. Thus, a path may be defined such that all of the rays actuating the apparatus must pass through an absorber interposed between successive counter trays. From observations of the counting-rate of the telescope as a function of the thickness and nature of the interposed absorber, the desired information is obtained.

In practice, owing to the finite aperture of the counter train, it is evident that the cosmic-ray particles detected in a particular absorption experiment may traverse total paths varying in length from L, the perpendicular separation between extreme trays, to $(L^2 + l^2 + w^2)^{\frac{1}{2}}$, the diagonal distance between opposite corners. For an accurate analysis of the results, it is thus necessary to determine the average path length through the telescope with respect to all of the radiation concerned, in order to obtain a factor for converting the measured thickness of the interposed material to effective thickness as regards absorption properties. The necessity for applying this correction has only rarely been appreciated (**1**),[3] and, as the following results indicate, the contribution is not always negligible. The error may become especially serious in the comparison of results obtained with different counter trains.

The effective path length must of course be calculated with respect to a particular dependence upon zenith angle θ of the unidirectional cosmic-ray intensity, $I(\theta)$, defined as the number of ionizing particles per square centimeter per unit solid angle per second arriving in a given direction. It has been established for some time that, at sea level, the variation is in good accord with the equation (**2,3,4,5**)

$$I(\theta) = I(0) \cos^2 \theta. \tag{1}$$

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[3] The boldface numbers in parentheses refer to the references appended to this paper.

At an altitude of 4300 m. approximately the same law (1) applies to high energy mesons having ranges exceeding 15 cm. of Pb, whereas for the soft component the exponent may be 3. A departure from this relationship apparently occurs at higher altitudes, as was shown by Swann, Locher and Danforth (6), who obtained a value of 0.21 for the ratio $I\,(90°)/I(0°)$.

In order to compare absorption measurements made in the stratosphere with those obtained at low altitudes, it is evidently important to ascertain, for the particular cosmic-ray telescopes utilized in obtaining the data, the magnitude of the dependence of the effective path length upon the nature of the angular distribution of cosmic-ray intensity. For this purpose, calculations have been based upon several different laws in addition to (1), representing alternative possible modes for the variation of intensity with zenith angle. At extremely high altitudes, such as have been attained by apparatus carried aloft in free

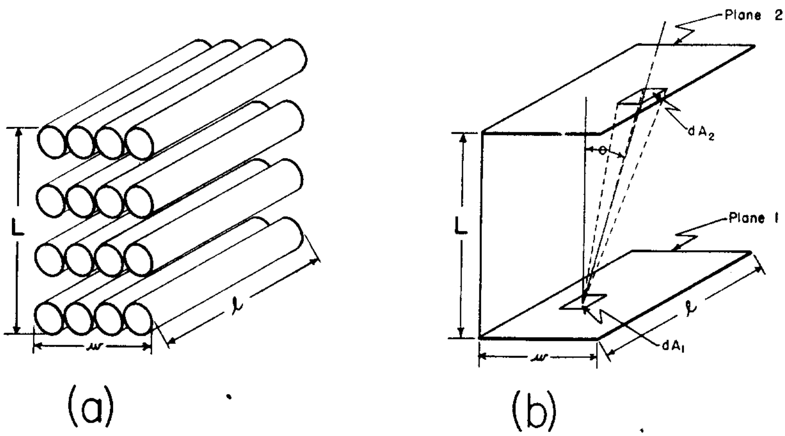


FIG. 1.

balloons during investigations conducted by one of the authors (7), the correct law probably lies within the range considered. Formulas have have been obtained, and families of curves have been computed showing the average path length through a cosmic-ray telescope as a function of all of the dimensions of the apparatus (8). These results may be utilized for obtaining corrections to be applied to the previously reported data of numerous investigators, or for designing systems for experiments at high altitudes in a manner such as to minimize the effects of changes in the directional characteristics of the cosmic radiation.

II. MATHEMATICAL DERIVATIONS.

A. Theory.

The trays in this cosmic-ray telescope are assumed to be rectangular and of length l and width w. The vertical distance between the uppermost and lowest trays is L. As shown in Fig. 1 the x axis is taken to coincide with one of the long edges of the lowest tray, and the z axis is vertical. The lowest tray is designated plane 1 and the cartesian coordinates of a point in it are $(x_1, y_1, 0)$. The uppermost tray is designated plane 2 and the coordinates of a point in it (x_2, y_2, L).

Consider two elements of area, dA_1 in the lowest tray and dA_2 in the uppermost tray. The cosmic rays which pass through both of these elements of area make an angle θ with the vertical. The number of cosmic rays which pass through both dA_1 and dA_2 per unit time is given by dN, where

$$dN = I(\theta) d\omega \, dA_1 \cos\theta, \tag{2}$$

where $d\omega$ is the solid angle which dA_2 subtends at dA_1. Obviously

$$d\omega = \frac{dA_2 \cos\theta}{\lambda^2}, \tag{3}$$

where λ is the distance from dA_2 to dA_1.

$I(\theta)$ is a constant for any given direction, but will depend upon the zenith angle θ, if the cosmic-ray intensity varies with θ. Substituting Eq. 3 in Eq. 2,

$$dN = \frac{I(\theta) dA_1 dA_2 \cos^2\theta}{\lambda^2}. \tag{4}$$

The most common formula given for the dependence of cosmic-ray intensity on zenith angle is represented by Eq. 1.

We will assume

$$I(\theta) = I(0)\cos^p\theta. \tag{1a}$$

Then

$$dN = \frac{I(0) dA_1 dA_2 \cos^{p+2}\theta}{\lambda^2}. \tag{5}$$

But from Fig. 1

$$\cos\theta = \frac{L}{\lambda}. \tag{6}$$

Hence,

$$dN = \frac{I(0) L^{p+2} dA_1 dA_2}{\lambda^{p+4}}. \tag{7}$$

Also

$$\lambda^2 = L^2 + (y_2 - y_1)^2 + (x_2 - x_1)^2 \tag{8}$$

and

$$\left.\begin{aligned} dA_1 &= dx_1 dy_1 \\ dA_2 &= dx_2 dy_2. \end{aligned}\right\} \tag{9}$$

If $\bar{\lambda}$ is the average path length through the cosmic-ray telescope, obviously

$$\bar{\lambda} = \frac{\int \lambda dN}{\int dN}. \tag{10}$$

This gives

$$\bar{\lambda} = \frac{\int_0^w dy_1 \int_0^w dy_2 \int_0^l dx_1 \int_0^l \frac{dx_2}{\lambda^{p+3}}}{\int_0^w dy_1 \int_0^w dy_2 \int_0^l dx_1 \int_0^l \frac{dx_2}{\lambda^{p+4}}}. \tag{11}$$

For any value of p the problem consists in evaluating integrals of the type

$$C_n \equiv \int_0^w dy_1 \int_0^w dy_2 \int_0^l dx_1 \int_0^l \frac{dx_2}{\lambda^n}. \tag{12}$$

We consider the cases when

$p = 2, 1, 0,$ and -1, so that
$n = 2, 3, 4, 5,$ and 6.

B. Two-Dimensional Problem ($w \ll 1$).

In the special case when w is so small that variation in y can be neglected

$$\bar{\lambda} = \frac{A_{p+3}}{A_{p+4}}, \tag{13}$$

where

$$A_n \equiv \int_0^l dx_1 \int_0^l \frac{dx_2}{[L^2 + (x_2 - x_1)^2]^{n/2}}. \tag{14}$$

Even when dealing with the general case where the variation in y cannot be neglected, it is worth while evaluating the A_n first, because by merely substituting $\sqrt{L^2 + (y_2 - y_1)^2}$ for L in these results one has the result of carrying out the two integrations in x_1 and x_2 in Eq. 12. Furthermore, the work of evaluating A_n is so easy for all n considered here that it is not necessary to give the details of the process of integration but merely the results as follows:

$$A_2 = \frac{2l}{L} \tan^{-1} \frac{l}{L} - \log \frac{L^2 + l^2}{L^2}, \tag{15}$$

$$A_3 = \frac{2}{L^2} \sqrt{L^2 + l^2} - \frac{2}{L}, \tag{16}$$

$$A_4 = \frac{l}{L^3} \tan^{-1} \frac{l}{L}, \tag{17}$$

$$A_5 = \frac{4}{3L^4}\sqrt{L^2 + l^2} - \frac{2}{3L^3} - \frac{2}{3L^2\sqrt{L^2 + l^2}}, \tag{18}$$

$$A_6 = \frac{1}{4L^4} - \frac{1}{4L^2(L^2 + l^2)} + \frac{3l}{4L^5}\tan^{-1}\frac{l}{L}. \tag{19}$$

It is desirable for both theoretical and practical reasons to write the formulas for $\bar{\lambda}$ in dimensionless form. For this reason we introduce $B_n(u)$, where

$$B_n(u) \equiv L^{n-2}A_n \tag{20}$$

and

$$u \equiv \frac{l}{L}. \tag{21}$$

It follows that

$$B_2(u) = 2u\tan^{-1}u - \log(1 + u^2), \tag{22}$$

$$B_3(u) = 2\sqrt{1 + u^2} - 2, \tag{23}$$

$$B_4(u) = u\tan^{-1}u, \tag{24}$$

$$B_5(u) = \frac{4}{3}\sqrt{1 + u^2} - \frac{2}{3(1 + u^2)} - \frac{2}{3}, \tag{25}$$

$$B_6(u) = \frac{3}{4}u\tan^{-1}u - \frac{1}{4(1 + u^2)} + \frac{1}{4}. \tag{26}$$

Substituting Eq. 20 in Eq. 13,

$$\frac{\bar{\lambda}}{L} = \frac{B_{p+3}(u)}{B_{p+4}(u)}, \tag{27}$$

where, as indicated in Eq. 1a, p determines the dependence of the intensity on zenith angle.

1. Series Approximations when $u < 1$.

$p = 2$ *($\cos^2\theta$ distribution):*

$$\frac{\bar{\lambda}}{L} = 1 + \frac{1}{12}u^2 - \frac{1}{15}u^4 + \frac{377}{6720}u^6 - \frac{29}{600}u^8 + \cdots. \tag{28}$$

$p = 1$ *($\cos\theta$ distribution):*

$$\frac{\bar{\lambda}}{L} = 1 + \frac{1}{12}u^2 - \frac{41}{720}u^4 + \frac{263}{6048}u^6 - \cdots. \tag{29}$$

$p = 0$ *(isotropic distribution):*

$$\frac{\bar{\lambda}}{L} = 1 + \frac{1}{12}u^2 - \frac{17}{360}u^4 + \frac{391}{12\,096}u^6 - \frac{44\,089}{1\,814\,400}u^8 + \cdots. \tag{30}$$

$p = -1$ *(sec. θ distribution)*:

$$\frac{\bar{\lambda}}{L} = 1 + \frac{1}{12}u^2 - \frac{3}{80}u^4 + \frac{19}{840}u^6 - \cdots. \tag{31}$$

All these series formulas are approximations valid only when $u < 1$.

It is interesting to note that for all the values of p considered:

$$\frac{\bar{\lambda}}{L} = 1 + \frac{1}{12}u^2 + \cdots \tag{32}$$

to that degree of approximation.

A study of the coefficients in formulas 28 to 31 shows that the numerical values obtained with the different distribution laws differ only slightly except when u is large. This is in accordance with the results of a qualitative approach to this problem.

C. Three-Dimensional Problem.

1. Evaluation of C_n.

We will now evaluate the integrals C_n defined in Eq. 12.

$n = 2$:

From Eqs. 12 and 15

$$C_2 = 2l\int_0^w dy_1\int_0^w \frac{1}{\sqrt{L^2+(y_2-y_1)^2}}\tan^{-1}\frac{l}{\sqrt{L^2+(y_2-y_1)^2}}dy_2$$
$$-\int_0^w dy_1\int_0^w \log\,[L^2+l^2+(y_2-y_1)^2]dy_2$$
$$+\int_0^w dy_1\int_0^w \log\,[L^2+(y_2-y_1)^2]dy_2. \tag{33}$$

The only one of the three integrals on the right that offers any difficulty is the first. It is evaluated by making the substitution:

$$\tan^{-1}\frac{l}{\sqrt{L^2+(y_2-y_1)^2}} = \sin^{-1}\frac{l}{\sqrt{L^2+l^2+(y_2-y_1)^2}}. \tag{34}$$

Then

$$\int_0^w dy_1\int_0^w \frac{1}{\sqrt{L^2+(y_2-y_1)^2}}\tan^{-1}\frac{l}{\sqrt{L^2+(y_2-y_1)^2}}dy_2$$
$$= \int_0^w dy_1\int_0^w \frac{1}{\sqrt{L^2+l^2+(y_2-y_1)^2}}$$
$$\times\left[1-\frac{l^2}{L^2+l^2+(y_2-y_1)^2}\right]^{-\frac{1}{2}}\sin^{-1}\frac{l}{\sqrt{L^2+l^2+(y_2-y_1)^2}}dy_2$$
$$= \sum_{r=0}^{\infty}\left\{\sum_{j=0}^{r}\frac{(2r-2j)!}{[(r-j)!]^2}\cdot\frac{(2j)!}{(j!)^2}\cdot\frac{1}{(2j+1)}\right\}\frac{l^{2r+1}}{2^{2r}}G_r', \tag{35}$$

where

$$G_r' \equiv \int_0^w dy_1 \int_0^w \frac{dy_2}{[L^2 + l^2 + (y_2 - y_1)^2]^{r+1}}. \tag{36}$$

This result is obtained by expanding the $\sin^{-1}$ and the square bracket in the integrand. The integration (9) of Eq. 36 yields:

$$G_0' = \frac{2w}{\sqrt{L^2 + l^2}} \tan^{-1} \frac{w}{\sqrt{L^2 + l^2}} - \log \frac{L^2 + l^2 + w^2}{L^2 + l^2}, \tag{37}$$

$$G_1' = \frac{w}{(L^2 + l^2)^{\frac{1}{2}}} \tan^{-1} \frac{w}{\sqrt{L^2 + l^2}} \tag{37a}$$

and for $r \geqslant 2$

$$\begin{aligned} G_r' \equiv {} & \frac{1}{(L^2 + l^2)^r} \sum_{\sigma=1}^{r-1} \frac{(2r)!}{(2r - 2\sigma + 1)!} \frac{[(r - \sigma)!]^2}{2^{2\sigma}(r!)^2} \cdot \frac{1}{(r - \sigma)} \\ & - \sum_{\sigma=1}^{r-1} \frac{(2r)!}{(2r - 2\sigma + 1)!} \frac{[(r - \sigma)!]^2}{2^{2\sigma}(r!)^2} \cdot \frac{1}{(r - \sigma)} \frac{1}{(L^2 + l^2)^\sigma} \\ & \times \frac{1}{(L^2 + l^2 + w^2)^{r-\sigma}} + \frac{2(2r)!}{2^{2r}(r!)^2} \cdot \frac{w}{(L^2 + l^2)^{r+\frac{1}{2}}} \tan^{-1} \frac{w}{\sqrt{L^2 + l^2}}. \end{aligned} \tag{37b}$$

Then

$$\begin{aligned} C_2 = {} & (L^2 + l^2 - w^2) \log (L^2 + l^2 + w^2) - (L^2 - w^2) \log (L^2 + w^2) \\ & - (L^2 + l^2) \log (L^2 + l^2) + L^2 \log L^2 + 4wL \tan^{-1} \frac{w}{L} \\ & - 4w\sqrt{L^2 + l^2} \tan^{-1} \frac{w}{\sqrt{L^2 + l^2}} \\ & + \sum_{r=0}^{\infty} \left\{ \sum_{j=0}^{r} \frac{(2r - 2j)!}{[(r - j)!]^2} \cdot \frac{(2j)!}{(j!)^2} \cdot \frac{1}{(2y + 1)} \right\} \frac{l^{2r+2}}{2^{2r-1}} G_r'. \end{aligned} \tag{38}$$

$n = 3$:

From Eqs. 12 and 16:

$$C_3 = 2 \int_0^w dy_1 \int_0^w \frac{[L^2 + l^2 + (y_2 - y_1)^2]^{\frac{1}{2}}}{[L^2 + (y_2 - y_1)^2]} dy_2 - 2 \int_0^w dy_1 \int_0^w \frac{1}{[L^2 + (y_2 - y_1)^2]} dy_2. \tag{39}$$

The first integral is evaluated by expanding the numerator of the integrand by the binomial theorem taking $(L^2 + l^2)$ as the larger term.

This gives

$$2\int_0^w dy_1 \int_0^w \frac{[L^2 + l^2 + (y_2 - y_1)^2]^{\frac{1}{2}}}{[L^2 + (y_2 - y_1)^2]} dy_2$$

$$= 2\sqrt{L^2 + l^2} \int_0^w dy_1 \int_0^w \frac{dy_2}{[L^2 + (y_2 - y_1)^2]}$$

$$+ 4\sqrt{L^2 + l^2} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2^{2r-1}} \cdot \frac{(2r-2)!}{(r-1)!r!} \cdot \frac{H_r'}{(L^2 + l^2)^r}. \quad (40)$$

Here

$$2H_r' \equiv \int_0^w dy_1 \int_0^w \frac{(y_2 - y_1)^{2r} dy_2}{[L^2 + (y_2 - y_1)^2]}. \quad (41)$$

This integral can easily be evaluated by dividing the denominator into the numerator; there is a remainder term. This gives

$$H_r' = \sum_{\sigma=0}^{r-1} \frac{(-1)^\sigma L^{2\sigma} w^{2r-2\sigma}}{(2r - 2\sigma)(2r - 2\sigma - 1)}$$

$$+ (-1)^r L^{2r-1} w \tan^{-1}\frac{w}{L} - (-1)^r \tfrac{1}{2} L^{2r} \log \frac{L^2 + w^2}{L^2}. \quad (42)$$

The final result for C_3 is

$$C_3 = 4\sqrt{L^2 + w^2} - 4L - 2w \log \frac{\sqrt{L^2 + w^2} + w}{\sqrt{L^2 + w^2} - w}$$

$$- 2\sqrt{L^2 + l^2} \log \frac{L^2 + w^2}{L^2} + \frac{4w\sqrt{L^2 + l^2}}{L} \tan^{-1}\frac{w}{L}$$

$$+ 4\sqrt{L^2 + l^2} \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{2^{2r-1}} \cdot \frac{(2r-2)!}{(r-1)!r!} \cdot \frac{H_r'}{(L^2 + l^2)^r}. \quad (43)$$

$n = 4$:

Combining Eq. 12 and 17

$$C_4 \equiv l \int_0^w dy_1 \int_0^w \frac{1}{[L^2 + (y_2 - y_1)^2]^{\frac{1}{2}}} \tan^{-1} \frac{l}{[L^2 + (y_2 - y_1)^2]} dy_2. \quad (44)$$

In all the integrations for C_n the independent variable for the first integration should be $(y_2 - y_1)$, with y_1 constant, of course. In this case it is advantageous to use partial integration first in such a way as to get rid of the $\tan^{-1}$ factor by differentiation in the double integral. The first integration in the resulting double integral can be done by separating into partial fractions. *All* the integrands for the integration with respect to y_1 involve a $\tan^{-1}$ term. Once more partial integration can be used in each integral to eliminate the $\tan^{-1}$ term in the resulting integral.

In this way one obtains

$$C_4 = \frac{2l\sqrt{L^2+w^2}}{L^2}\tan^{-1}\frac{l}{\sqrt{L^2+w^2}} + \frac{2w\sqrt{L^2+l^2}}{L^2}\tan^{-1}\frac{w}{\sqrt{L^2+l^2}} - \frac{2l}{L}\tan^{-1}\frac{l}{L} - \frac{2w}{L}\tan^{-1}\frac{w}{L} - \log\frac{L^2(L^2+l^2+w^2)}{(L^2+l^2)(L^2+w^2)}. \quad (45)$$

$n = 5$:

Using Eqs. 12 and 18

$$C_5 = \frac{4}{3}\int_0^w dy_1\int_0^w \frac{[L^2+l^2+(y_2-y_1)^2]^{\frac{1}{2}}}{[L^2+(y_2-y_1)^2]}dy_2 - \frac{2}{3}\int_0^w dy_1\int_0^w \frac{dy_2}{[L^2+(y_2-y_1)^2][L^2+l^2+(y_2-y_1)^2]^{\frac{1}{2}}} - \frac{2}{3}\int_0^w dy_1\int_0^w \frac{dy_2}{[L^2+(y_2-y_1)^2]^{\frac{1}{2}}}. \quad (46)$$

In the case of the first two integrals the factors in the integrands with the exponent $\frac{1}{2}$ are expanded by means of the binomial theorem in such a way that the positive powers of $(y_2 - y_1)$ appear in all terms except the first. It is exactly the same procedure as that used in evaluating C_3. Therefore the details will not be given. The result is

$$C_5 = \frac{4}{3L} - \frac{4\sqrt{L^2+w^2}}{3L^2} + \frac{4w\sqrt{L^2+l^2}}{3L^3}\tan^{-1}\frac{w}{L} + \frac{2}{3}\sum_{r=1}^{\infty}\frac{(-1)^r}{2^{2r-2}}\cdot\frac{(2r-2)!}{(r-1)!r!}(L^2+l^2)^{-r+\frac{1}{2}}J_r'. \quad (47)$$

Here

$$J_r' = \sum_{\sigma=1}^{r-1}(-1)^{\sigma-1}\frac{w^{2r-2\sigma}L^{2\sigma-2}}{(2r-2\sigma-1)} + (-1)^{r-1}L^{2r-3}w\tan^{-1}\frac{w}{L}. \quad (48)$$

$n = 6$:

From Eqs. 12 and 19

$$C_6 = \frac{1}{4}\int_0^w dy_1\int_0^w \frac{dy_2}{[L^2+(y_2-y_1)^2]^2} - \frac{1}{4}\int_0^w dy_1\int_0^w \frac{dy_2}{[L^2+l^2+(y_2-y_1)^2][L^2+(y_2-y_1)^2]} + \frac{3l}{4}\int_0^w dy_1\int_0^w \frac{1}{[L^2+(y_2-y_1)^2]^{5/2}}\tan^{-1}\frac{l}{\sqrt{L^2+(y_2-y_1)^2}}dy_2. \quad (49)$$

The second and third integrals are evaluated by the same methods as in the case of C_4, namely by partial integration on integrands involving $\tan^{-1}$ so as to get rid of $\tan^{-1}$ in the resulting *integral* term, and separation in partial fractions whenever integrands such as those in the second integral appear.

The result is

$$C_6 = \frac{(2l^2 + L^2)w}{2L^4\sqrt{L^2 + l^2}} \tan^{-1} \frac{w}{\sqrt{L^2 + l^2}} + \frac{(2w^2 + L^2)l}{2L^4\sqrt{L^2 + w^2}} \tan^{-1} \frac{l}{\sqrt{L^2 + w^2}} - \frac{w}{2L^3} \tan^{-1} \frac{w}{L} - \frac{l}{2L^3} \tan^{-1} \frac{l}{L}. \quad (50)$$

The infinite series in C_2 and C_3 are convergent for all values of the parameters L, l, and w; and the infinite series in C_5 is convergent provided $w \leqslant l$, a condition which can be satisfied in all cases.

2. Reduction to Dimensionless Form.

Equation 11 may be rewritten as

$$\bar{\lambda} = \frac{C_{p+3}}{C_{p+4}}. \quad (51)$$

For both theoretical and practical reasons it is desirable to write this equation in dimensionless form.

Let

$$D_n(u, v) = L^{n-4}C_n, \quad (52)$$

$$u \equiv \frac{l}{L}, \quad (21)$$

$$v \equiv \frac{w}{L}. \quad (53)$$

Then D_n is dimensionless.

From Eqs. 51 and 52

$$\frac{\bar{\lambda}}{L} = \frac{D_{p+3}(u, v)}{D_{p+4}(u, v)}. \quad (54)$$

This dimensionless equation gives $\bar{\lambda}/L$ for the cosmic-ray distribution determined by the value of p.

Below, the formulas for $D_n(u, v)$ are given:

$n = 2$:

$$D_2(u, v) = \log \frac{1 + u^2 + v^2}{(1 + u^2)(1 + v^2)} + u^2 \log \frac{1 + u^2 + v^2}{1 + u^2} + v^2 \log \frac{1 + v^2}{1 + u^2 + v^2} + 4v \tan^{-1} v - 4v\sqrt{1 + u^2} \tan^{-1} \frac{v}{\sqrt{1 + u^2}} + \sum_{r=0}^{\infty} g_r G_r u^{2r+2}, \quad (55)$$

where

$$g_r = \frac{1}{2^{2r-1}} \sum_{j=0}^{r} \frac{(2r-2j)!}{[(r-j)!]^2} \frac{(2j)!}{[j!]^2} \cdot \frac{1}{(2j+1)} . \tag{56}$$

and

$$G_0 = \frac{2v}{\sqrt{1+u^2}} \tan^{-1} \frac{v}{\sqrt{1+u^2}} - \log \frac{1+u^2+v^2}{1+u^2} , \tag{57}$$

$$G_1 = \frac{v}{(1+u^2)^{\frac{3}{2}}} \tan^{-1} \frac{v}{\sqrt{1+u^2}} , \tag{57a}$$

and for $r \geqslant 2$

$$\begin{aligned} G_r = {} & \frac{1}{(1+u^2)^r} \sum_{\sigma=1}^{r-1} \frac{(2r)![(r-\sigma)!]^2}{2^{2\sigma}(r!)^2(2r-2\sigma+1)!} \cdot \frac{1}{(r-\sigma)} \\ & - \sum_{\sigma=1}^{r-1} \frac{(2r)![(r-\sigma)!]^2}{2^{2\sigma}(r!)^2(2r-2\sigma+1)!} \cdot \frac{1}{(r-\sigma)} \frac{1}{(1+u^2)^{\sigma}(1+u^2+v^2)^{r-\sigma}} \\ & + \frac{(2r)!}{2^{2r}(r!)^2} \cdot \frac{v}{(1+u^2)^{r+\frac{1}{2}}} \tan^{-1} \frac{v}{\sqrt{1+u^2}} . \end{aligned} \tag{57b}$$

$n = 3$:

$$\begin{aligned} D_3(u, v) = {} & 4\sqrt{1+v^2} - 4 - 2v \log \frac{\sqrt{1+v^2}+v}{\sqrt{1+v^2}-v} \\ & - 2\sqrt{1+u^2} \log (1+v^2) + 4v\sqrt{1+u^2} \tan^{-1} v \\ & - 8\sqrt{1+u^2} \sum_{r=1}^{\infty} \frac{(-1)^r(2r-2)!}{2^{2r}(r-1)!r!} (1+u^2)^{-r} H_r , \end{aligned} \tag{58}$$

where

$$\begin{aligned} H_r = {} & \sum_{\sigma=0}^{r-1} \frac{(-1)^{\sigma} v^{2r-2\sigma}}{(2r-2\sigma)(2r-2\sigma-1)} \\ & + (-1)^r v \tan^{-1} v - (-1)^r \tfrac{1}{2} \log (1+v^2) . \end{aligned} \tag{59}$$

$n = 4$:

$$\begin{aligned} D_4(u, v) = {} & 2u\sqrt{1+v^2} \tan^{-1} \frac{u}{\sqrt{1+v^2}} \\ & + 2v\sqrt{1+u^2} \tan^{-1} \frac{v}{\sqrt{1+u^2}} - 2u \tan^{-1} u \\ & - 2v \tan^{-1} v - \log \frac{1+u^2+v^2}{(1+u^2)(1+v^2)} . \end{aligned} \tag{60}$$

$n = 5$:

$$D_5(u, v) = \frac{4}{3} - \frac{4}{3}\sqrt{1 + v^2} + \frac{4}{3} v\sqrt{1 + u^2} \tan^{-1} v$$

$$+ \frac{8}{3} \sum_{r=1}^{\infty} \frac{(-1)^r (2r - 2)!}{2^{2r}(r - 1)!r!} (1 + u^2)^{-r+\frac{1}{2}} J_r, \quad (61)$$

where

$$J_r = \sum_{\sigma=1}^{r-1} (-1)^{\sigma-1} \frac{v^{2r-2\sigma}}{2r - 2\sigma - 1} + (-1)^{r-1} v \tan^{-1} v. \quad (62)$$

$n = 6$:

$$D_6(u, v) = \frac{(2u^2 + 1)v}{2\sqrt{1 + u^2}} \tan^{-1} \frac{v}{\sqrt{1 + u^2}}$$

$$+ \frac{(2v^2 + 1)u}{2\sqrt{1 + v^2}} \tan^{-1} \frac{u}{\sqrt{1 + v^2}}$$

$$- \tfrac{1}{2} v \tan^{-1} v - \tfrac{1}{2} u \tan^{-1} u. \quad (63)$$

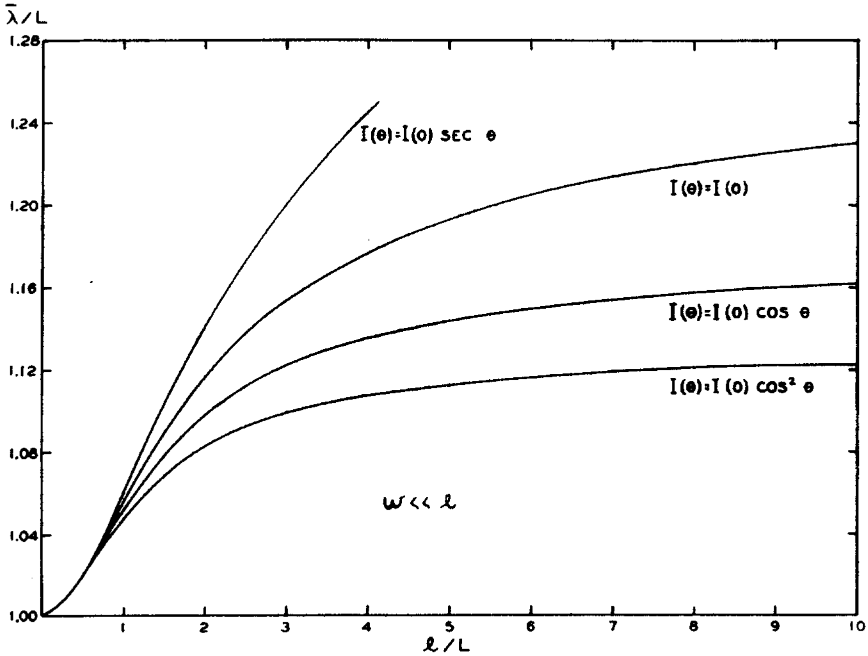
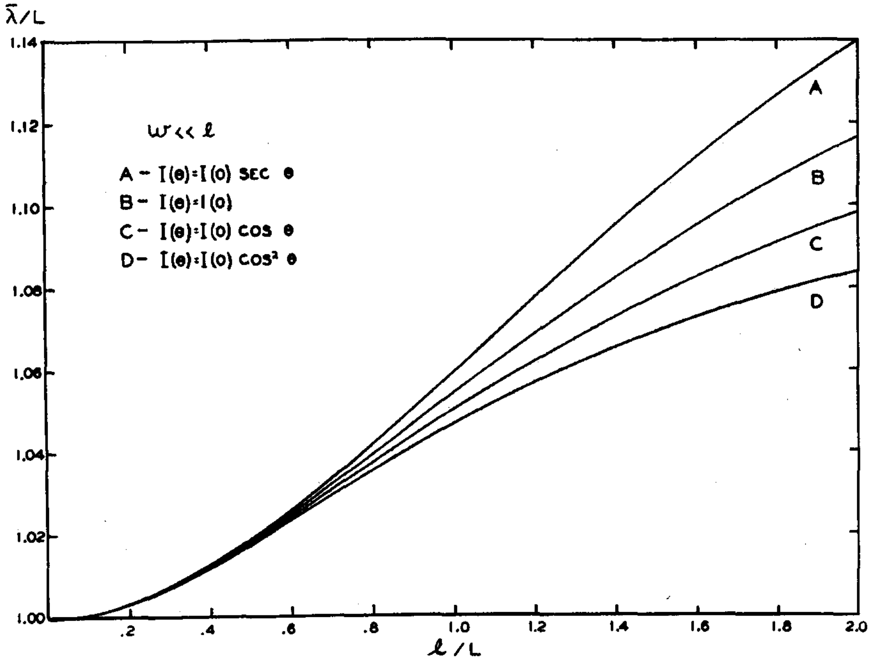


FIG. 2.

III. RESULTS.

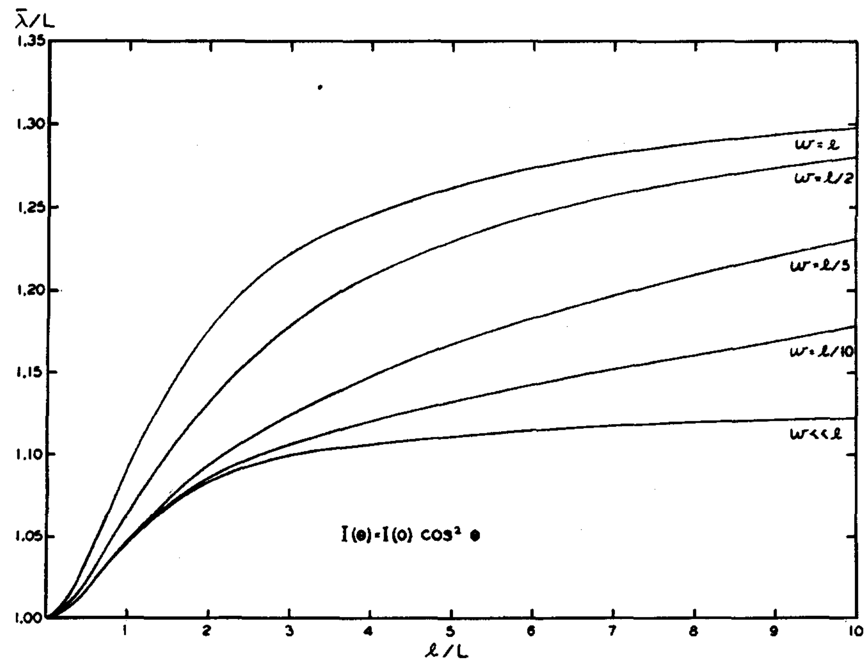
The results obtained from numerical computations utilizing the formulas developed in Section II are best presented in a series of graphs.

Figure 2 shows $\bar{\lambda}/L$ (the ratio of the average path length to the normal path length) plotted as a function of l/L (the ratio of counter



λ̄/L
1.14
1.12
1.10
1.08
1.06
1.04
1.02
1.00
w << l
A – I(θ) = I(0) SEC θ
B – I(θ) = I(0)
C – I(θ) = I(0) COS θ
D – I(θ) = I(0) COS² θ
A
B
C
D
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
l / L

FIG. 3.



λ̄/L
1.35
1.30
1.25
1.20
1.15
1.10
1.05
1.00
w = l
w = l/2
w = l/5
w = l/10
w << l
I(θ) = I(0) cos² θ
1
2
3
4
5
6
7
8
9
10
l / L

FIG. 4.

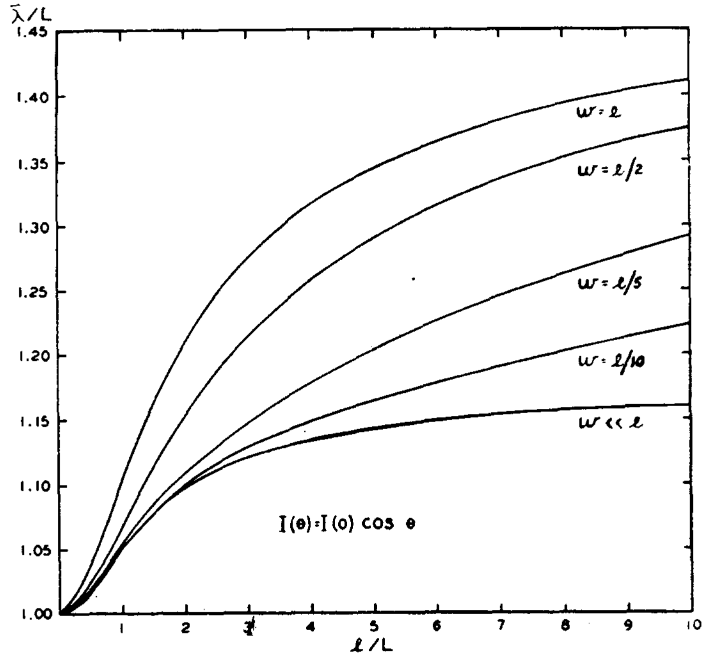

$\bar{\lambda}/L$
1.45
1.40
1.35
1.30
1.25
1.20
1.15
1.10
1.05
1.00
w = ℓ
w = ℓ/2
w = ℓ/5
w = ℓ/10
w << ℓ
I(θ) = I(0) cos θ
1
2
3
4
5
6
7
8
9
10
ℓ/L

FIG. 5.

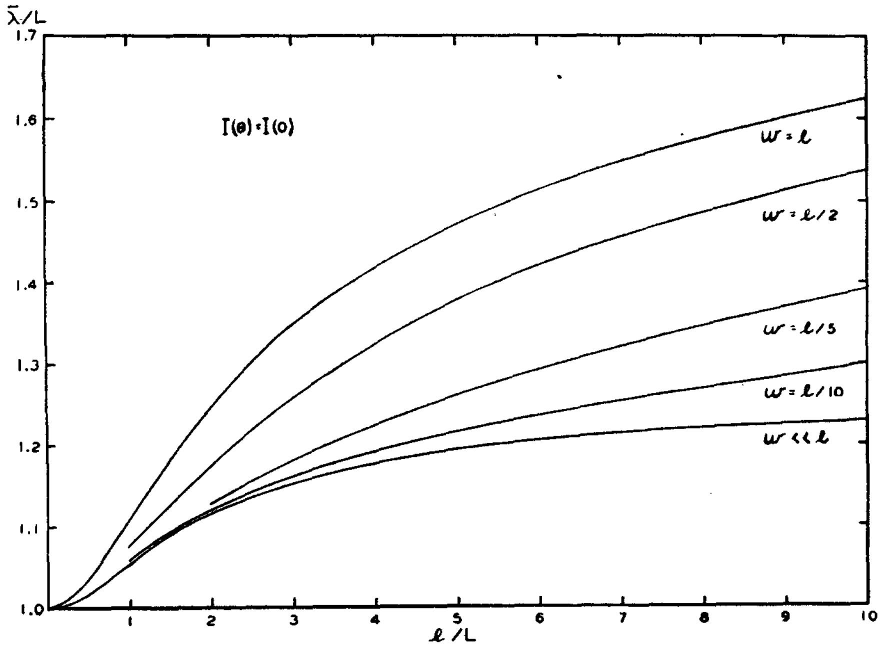
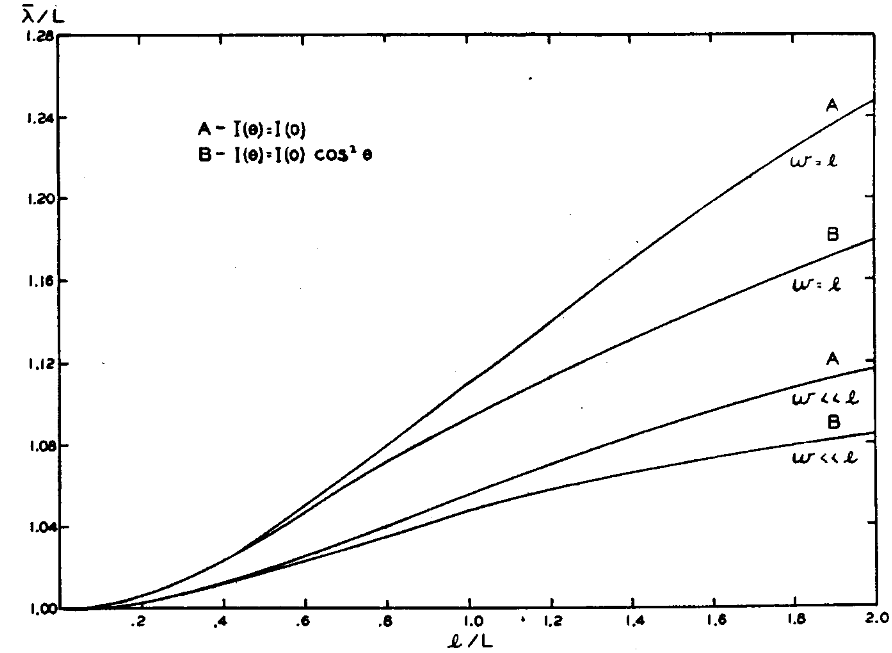
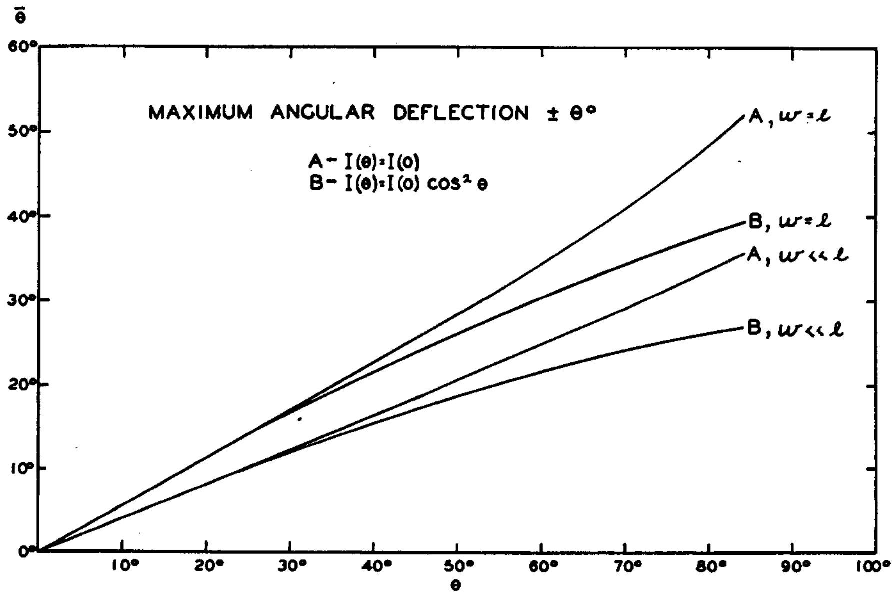

$\bar{\lambda}/L$
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
I(θ) = I(0)
w = ℓ
w = ℓ/2
w = ℓ/5
w = ℓ/10
w << ℓ
1
2
3
4
5
6
7
8
9
10
ℓ/L

FIG. 6.



λ̄/L
1.28
1.24
1.20
1.16
1.12
1.08
1.04
1.00
A - I(θ)=I(0)
B - I(θ)=I(0) cos² θ
A
w=l
B
w=l
A
w<<l
B
w<<l
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
l/L

FIG. 7.



θ̄
60°
50°
40°
30°
20°
10°
0°
MAXIMUM ANGULAR DEFLECTION ± θ°
A - I(θ)=I(0)
B - I(θ)=I(0) cos² θ
A, w=l
B, w=l
A, w<<l
B, w<<l
10°
20°
30°
40°
50°
60°
70°
80°
90°
100°
θ

FIG. 8.

length to the vertical separation between extreme counters) for various indicated laws of cosmic-ray intensity *versus* zenith angle. These curves, as well as those plotted in Fig. 3 on an expanded scale encompassing a range of dimensions more commonly encountered in the usual experimental arrangements, apply when the counter diameter is negligible compared with its length.

Figure 4 contains a family of curves relating the same quantities for a $\cos^2\theta$ distribution of intensity. Here, the width of the tray is taken into account. The magnitude of w expressed in terms of counter length appears as the parameter indicated in Figs. 5 and 6 which represent $\cos\theta$ and isotropic distributions in a similar manner. The extreme cases of $w = l$ and $w \ll l$ for both the isotropic and $\cos^2\theta$ distributions are shown on an expanded scale in Fig. 7.

It is of interest to plot the results in terms of angles, as has been done in Fig. 8. The so-called angular resolution of a vertical cosmic-ray telescope is defined as the maximum allowed zenith angle for a ray which can produce a coincidence. This angle $\theta_R = \tan^{-1} l/L$. The angle between the vertical and the average path through the telescope is $\bar{\theta} = \sec^{-1} \bar{\lambda}/L$. This greatly simplifies the shape of the curves, as may be seen in Fig. 8. For small angles, the curves are practically straight lines. Thus, the results are conveniently summarized by the following simple expression

$$\bar{\theta} = k_1\theta_R. \tag{64}$$

Values of k_1 and the condition under which they pertain, are tabulated in Table I.

TABLE I.

Zenith Angle Distribution Law	k_1	Range of θ_R	w
$I(\theta) = I(0)\cos^2\theta$	0.40	0°–25°	$\ll l$
	0.36	25°–40°	
	0.33	40°–55°	
	0.55	0°–30°	$= l$
	0.47	30°–60°	
	0.37	60°–80°	
$I(\theta) = I(0)$	0.41	0°–45°	$\ll l$
	0.43	45°–80°	
	0.55	0°–30°	$= l$
	0.60	30°–60°	
	0.68	60°–80°	

IV. CONCLUSIONS.

These derivations and calculations have been performed only for arrangements in which the dimensions of the uppermost tray are the same as those for the lowest tray. Other cases concerning trays of different sizes could of course be treated in a manner similar to that used above. If the dimensions of extreme trays are not very different, the use of arithmetical means of the corresponding dimensions should yield results which are sufficiently accurate for most practical purposes. The most striking aspects of the results presented quantitatively in the various figures may be summarized qualitatively as follows: (1) the average path length through a vertical coincidence counter train is not greatly affected by a change in the zenith angle distribution law, provided the separation between extreme trays is roughly comparable with the dimensions of the trays; (2) even for geometrical arrangements which intuitively appear to be unsatisfactory as regards angular definition and the possible complicating effects arising from the distribution of allowed path lengths, the departure of the average path length from the vertical path length may be rather small.

V. ACKNOWLEDGMENTS.

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